Non-Local Product Rules for Percolation

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Despite original claims of a first–order transition in the product rule model proposed by Achlioptas *et al.* [*Science* **323**, 1453 (2009)], recent studies indicate that this percolation model, in fact, displays a continuous transition. The distinctive scaling properties of the model at criticality, however, strongly suggest that it should belong to a different universality class than ordinary percolation. Here we introduce a generalization of the product rule that reveals the effect of non–locality on the critical behavior of the percolation process. Precisely, pairs of unoccupied bonds are chosen according to a probability that decays as a power-law of their Manhattan distance, and only that bond connecting clusters whose product of their sizes is the smallest, becomes occupied. Interestingly, our results for two-dimensional lattices at criticality shows that the power-law exponent of the product rule has a significant influence on the finite-size scaling exponents for the spanning cluster, the conducting backbone, and the cutting bonds of the system. In all three cases, we observe a continuous variation from ordinary to (non-local) explosive percolation exponents.

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The percolation paradigm represents a formidable example where a simple geometrical construction leads to profound concepts in statistical physics, with special emphasis on phase transitions, and real applications in science and technology [1–3]. Standard percolation processes are based on local rules, since they are accomplished through random allocation of sites or bonds, therefore disregarding any spatial correlation or global information involved in the occupation of other elements on the lattice. However, in the case of long-range spatially-correlated percolation [4, 5], the probability for a site to be occupied depends on the occupancy of other sites. Moreover, it has been shown that spatial long-range correlations in site occupancy can give rise to important changes on the structural characteristics of the spanning cluster as well as its corresponding conducting backbone [5]. These changes are strong enough to modify the scaling exponents of traditional (local) percolation.

Recently, a new percolation model has been proposed, the so-called Product Rule (PR) percolation, in terms of a bond occupation process that is essentially non-local [6]. In this model, at each step, two unoccupied bonds are randomly chosen and associated with weights given by the product of the cluster sizes they would potentially connect. Only that bond which has the smallest weight becomes occupied. By comparison with the traditional percolation model [1], the PR model presents a more abrupt transition when applied to different network topologies [7–14]. As potential applications, the PR model has been recently associated to the growth dynamics of Protein Homology Networks [15] as well as to the formation of bundles of single-walled nanotubes [16].

Regardless of initial claims of a first order transition in the PR model [6], however, recent analytical and numerical works [17, 18] have demonstrated that the alleged "Explosive Percolation" process actually displays a continuous, i.e., a second-order phase transition. This

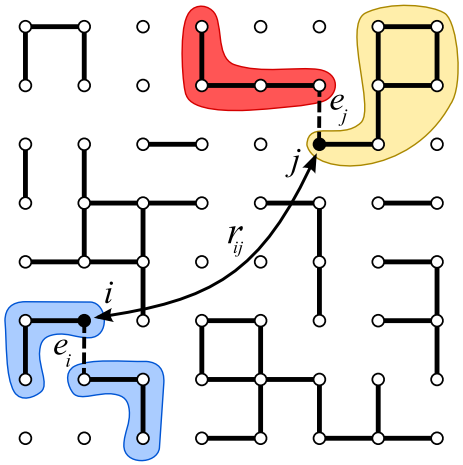

FIG. 1: (Color online) Pair of unoccupied bonds e_i and e_j (dashed lines) randomly selected for the application of the product rule, according to the probability $P(r_i) \sim r_j^{-\alpha}$, where r_{ij} is the Manhattan distance between sites i and j (black circles), and α is a variable exponent. Following the PR model, the bond e_i, merging the two clusters in blue (with 3 sites each), becomes occupied. The bond e_j would merge the the clusters in red (4 sites) and yellow (6 sites), but remains unoccupied.

apparent drawback of the PR model has been somehow overstated, in the sense that the model proposed by Achlioptas *et al.* [6] certainly represents an original and interesting contribution to the field. For instance, much less importance has been given to the non-local attributes of the PR algorithm. As a consequence of this non-locality, the model exhibits a percolation transition that, although continuous in nature, seems to belong to a different universality class than ordinary percolation [9, 12, 14]. Under this framework, the adjacent edge rule (AER) model [19] represents a particular case of the PR process that is analytically tractable, since the selection

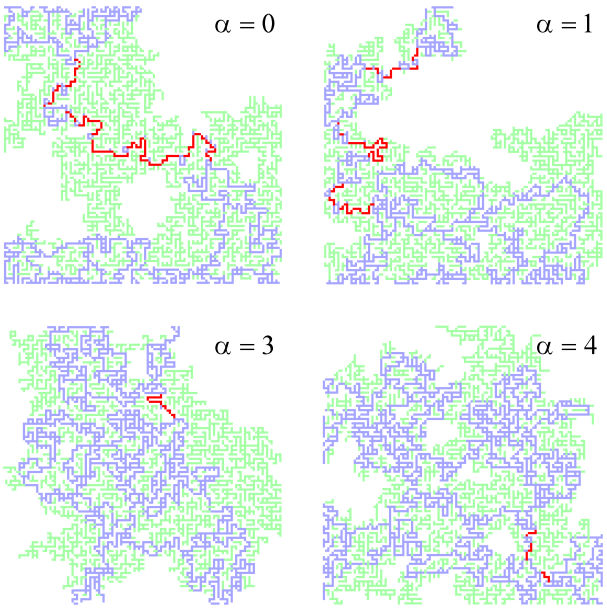

FIG. 2: (Color online) Snapshots of the largest cluster at p_c for different values of the exponent α, and a lattice size $L = 64$. The bonds forming the conducting backbone are in blue, the cutting bones are in red, and the remaining bonds of the largest cluster are presented in green. Although no major difference can be observed on the mass M_{clus} of the largest cluster, one can notice that the conducting backbone occupies a larger fraction of the largest cluster as α increases, leading to a substantial decrease on the number of cutting bones M_{cut}.

is restricted to two adjacent bonds. When applied to random graphs, this model still displays a more abrupt transition and different scaling properties than ordinary percolation.

In this Letter we introduce a generalization of the PR model in which the range of its non-local features can be systematically controlled. This is carried out imposing that pairs of bonds for selection are randomly chosen according to a probability that decays as a power-law of their Manhattan distance, namely the distance measured as the number of connections separating the sites in a regular lattice. This physically plausible assumption is inspired on a geographical model for complex networks where long-ranged shortcuts are incorporated to regular lattices. Such a conceptual construction has been extensively used as a way to explain the emergence of optimal navigation and efficient transport in small-world systems [20]. As a consequence of the selection rule adopted here, we show that the scaling properties of the system becomes dependent on the specific value of the corresponding power-law exponent. A continuous variation is then revealed from the traditional to the PR percolation behavior. Moreover, the results of our extensive numerical simulations provide strong evidence for the fact that the AER model, when applied to regular lattices in two-dimensions, falls in the same universality class as ordinary percolation.

Our bond percolation process takes place on a square lattice of size L. At each step, two sites i and j are randomly selected with probability $P(r_{ij}) \sim r_{ij}^{-\alpha}$, where r_{ij} is the Manhattan distance between i and j, measured as the number of connections separating these sites in the underlying regular lattice [21]. From each site i and j, one bond is then selected among its four adjacent edges, namely e_i and e_j, respectively. If at least one of these two bonds is already occupied, the entire process of selection is restarted. If not, following the product rule, weights are assigned to each of these bonds, in proportionality to the product of the size (number of sites) of the clusters they would potentially connect. In the case a bond connects two sites in the same cluster, the weight is equal to the square of the cluster size. The bond associated with the smallest weight becomes occupied, while the other stays unoccupied, but can be selected again in later steps.

Our model displays two distinct limiting behaviors, depending on the exponent α. For $\alpha = 0$, we recover the usual PR, for which the preliminary random selection of the bonds e_1 and e_2 constitutes a highly non–local process [8, 17]. In the limit of $\alpha \to \infty$, the bonds e_1 and e_2 are always adjacent, which corresponds to the AER process proposed in Ref. [19], but applied here to regular lattices. Although more spatially restricted than the PR process, the AER is still non–local, since it requires information about the masses of the joining clusters [17]. As we show later, the finite low-dimensionality of the square lattice employed here attenuates even further the already weaker non–local features of the AER process.

The percolation process stops when one among all clusters, namely the spanning cluster, connects the lattice from top to bottom [1]. At that point, the fraction p of occupied bonds corresponds to the percolation threshold p_c. For $\alpha = 0$, we obtain $p_c = 0.527 \pm 0.001$, which is in good agreement with previous simulation results of the PR on the square lattice [11, 12]. Moreover, we find that p_c decreases smoothly and monotonically with α from this value to 0.522 ± 0.001 at $\alpha = 4$ (not shown). Next we apply the burning algorithm [22] to compute the mass of the spanning cluster M_{clus}, the mass of its conducting backbone M_{back}, and the mass (number) M_{cut} of cutting bonds. The last ones, also called red bonds, if removed, would break the spanning cluster in two, therefore destroying the global connectivity of the system.

Our results show that, regardless of the value of α, all these critical quantities scale with the system size L as typical power laws, $M_{back} \sim L^{d_{back}}$, $M_{cut} \sim L^{d_{cut}}$ (see Fig. 3), and $M_{clus} \sim L^{d_{clus}}$ (not shown), where d_{back}, d_{cut} and d_{clus} are the fractal dimensions of the conducting backbone, the cutting bonds, and the largest cluster, respectively. In Figs. 4(a)-(c) we show that all these exponents exhibit a monotonic variation with α, going

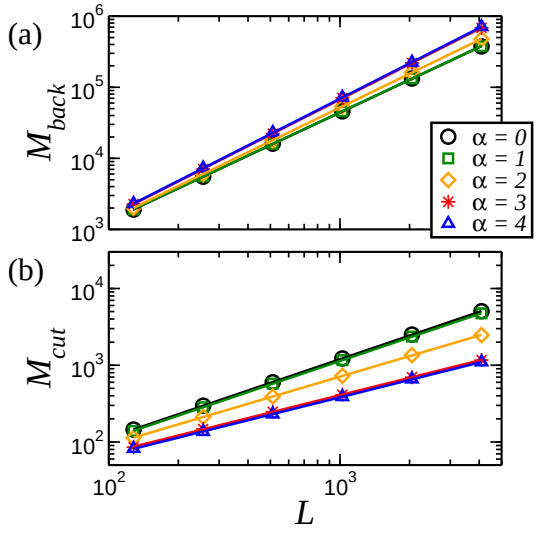

FIG. 3: (Color online) (a) Log-log dependence of the mass of the conducting backbone M_{back} on the system size for different values of the exponent α. (b) the same as in (a), but for the number of cutting bonds M_{cut}. In both cases and for all values of α, the evidence of scaling behavior substantiates the calculation of the fractal dimensions d_{back} and d_{cut} as the slopes of the corresponding straight lines that are best–fitted to the simulation data. All quantities are averaged over at least 2500 realizations precisely at the point in which the largest cluster appears.

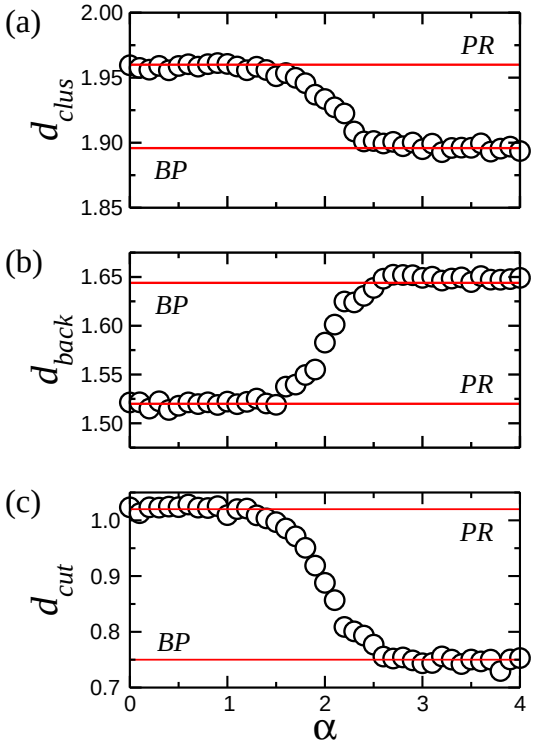

FIG. 4: (Color online) Dependence on the exponent α of the size-scaling exponents for (a) the mass of the spanning cluster d_{clus}, (b) the mass of the conducting backbone d_{back}, and (c) the number of cutting bonds d_{cut}. In all cases, a crossover can be observed in the interval $1 < \alpha < 3$ from a regime of non-local explosive percolation at $\alpha = 0$, to a regime that is compatible with ordinary bond percolation (BP), at sufficiently large values of α. The dashed red lines correspond to $d_{clus} = 1.96$ [9, 10] and 91/48 [1] in (a), $d_{back} = 1.52$ [14] and 1.64 [2] in (b), and $d_{cut} = 1.02$ [14] and 0.75 [1] in (c).

from a saturation regime of (non–local) explosive percolation at $\alpha = 0$ to another compatible with ordinary bond percolation (BP) at sufficiently large values of α. Accordingly, for $\alpha = 0$, we recover the previously numerically calculated values of $d_{clus} = 1.96 \pm 0.01$ [9, 10], $d_{back} = 1.52 \pm 0.03$, and $d_{cut} = 1.02$ [14]. In all three cases, by increasing α, a crossover from PR to BP takes place in the interval $1 < \alpha < 3$. More precisely, d_{clus} decreases in this interval and starts fluctuating around 1.89 for $\alpha > 3$ (see Fig. 4a), in agreement with the classical $2D$ value of 91/48 [1]. After increasing in the interval $1 < \alpha < 3$, the exponents d_{back} and d_{cut} remain practically constant for $\alpha > 3$, around 1.64 [2] (see Fig. 4b) and 0.75 [1] (see Fig. 4c), respectively. These values are fully compatible with previously reported numerical calculations for ordinary (local) percolation in $2D$. The variations of the exponents d_{clus} and d_{back} within $1 < \alpha < 3$ reflect relevant changes in compactness of the spanning cluster and its conducting backbone. Although the spanning cluster becomes less compact as α increases (d_{clus} decreases), the mass of the backbone M_{back} tends to occupy a larger fraction of M_{clus}, since the dimension d_{back} increases in the same interval of α values, as shown in Fig 3. In these circumstances, a more compact conducting backbone implies a smaller number of cutting bones (see Figure 3), therefore explaining the decrease in the exponent d_{cut}.

Next we provide some analytical arguments that indicate how non–local features are introduced in our percolation model through the power–law probability $P(r_{ij})$. We first consider the average distance between all pairs of sites in a empty lattice, $\langle r \rangle = \sum_{r=1}^{L} r N_r r^{-\alpha} / \sum_{r=1}^{L} N_r r^{-\alpha}$, where $N_r = 4r$ is the number of sites that are at a Manhattan distance r from a given site in the square lattice. Approximating the sum by an integral, we obtain that $\langle r \rangle \sim \int_1^L r^{2-\alpha} dr$. It follows that, for $\alpha < 2$, $\langle r \rangle$ is limited by the network size, leading to $\langle r \rangle \sim L$, while, for $2 \leq \alpha < 3$, this distance scales as $\langle r \rangle \sim L^{3-\alpha}$. For $\alpha \geq 3$ and sufficiently large lattice sizes, $\langle r \rangle$ is always finite. As a consequence, the effect of non–locality on the scaling properties of the system would only play a role for $\alpha < 3$. In addition, distinct non-local behaviors should be expected for the intervals $\alpha < 2$ and $2 \leq \alpha < 3$. These characteristics are consistent with the results displayed in Fig. 4. The observed mismatch between expected and numerically calculated crossover values of the exponent α is a consequence of finite-size scaling effects as well as the fact that the sequential bond allocation leads to the presence of spatial

D	$p_{c,AP}$	$p_{c,AER}$	d_{clus} - AP	d_{clus} - AER	d_{clus} - Classical [1]
2	0.526550 ± 0.000005	0.52007 ± 0.00001	1.955 ± 0.002	1.899 ± 0.001	91/48
3	0.322096 ± 0.000001	0.285360 ± 0.000008	2.788 ± 0.003	2.530 ± 0.003	2.53
4	0.234160 ± 0.000003	0.202163 ± 0.000004	3.665 ± 0.009	3.079 ± 0.005	3.06
5	0.184656 ± 0.000006	0.160454 ± 0.000004	4.61 ± 0.01	3.59 ± 0.04	3.54
6	0.152642 ± 0.000005	0.134113 ± 0.000002	5.558 ± 0.005	4.46 ± 0.01	4

TABLE I: Estimated values of the percolation threshold p_c and the scaling exponent d_{clus} for hyper-cubic lattices of dimension D calculated using the jump method [13, 23]. The presented values correspond to averages over a minimum of 2500 realizations of systems with sizes up to $L = 4096$ ($D = 2$), 256 ($D = 3$), 64 ($D = 4$), 28 ($D = 5$), and 16 ($D = 6$).

correlations in the percolation process. These correlations make the assumption of an ever empty lattice, as adopted to compute $\langle r \rangle$, no longer strictly valid.

In order to better confirm the role of non–locality on the PR process, we perform additional simulations in the two limits of the model at higher dimensions, namely for $\alpha = 0$, which corresponds to the original PR process, and for the AER process, $\alpha \to \infty$. In these cases, improved performance can be achieved by adopting the so-called jump method to analyze the behavior of the order parameter M_{clus} [13, 23]. For each realization, we compute the average fraction p of occupied bonds at which a jump takes place, defined as the maximum change on M_{clus} from the occupation of a single bond. This value of p corresponds to the percolation threshold p_c. The results for p_c and d_{clus} in both limits and different dimensions are summarized in Table I. Interestingly, the discrepancy between the fractal dimensions calculated for PR and AER models increases substantially with lattice dimensionality. Moreover, our calculations suggest that the resemblance between regular BP and the limiting case $\alpha \to \infty$ stands only up to five dimensions, when compared with previous results from the literature [1, 2].

In summary, we have proposed a generalization of the PR model where the range of non–locality in the percolation process can be explicitly tuned. Our results show that this model displays a rich variety of scaling behaviors, going from ordinary to non–local explosive percolation. We expect our PR model, since it is based on a geographical choice of bond pairs, to provide relevant physical insights into the role of non–locality on the critical properties of percolation.

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[21] In fact, to account for the number N_r of neighbors that are at a Manhattan distance r from a particular site in a square lattice ($N_r = 4r$), the random distance r must be generated following a power-law distribution proportional to $r^{(1-\alpha)}$.

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